Convergence of Markov Chain (Remaining Proof Explanation)

Infimum of Inverse Gamma

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We have given

$$\theta|\mu, y \sim IG\left(\frac{m-1}{2}, \frac{s^2 + m(\mu - \bar{y})^2}{2}\right)$$

We have to find $\inf_{(\mu', \theta^{'}) \in C} \pi(\theta|\mu^{'}, y)$ where C is

$$C = \{(\mu, \theta) : V(\mu, \theta) \leq d\}$$
$$where\ V(\mu, \theta) = (\mu - \bar{y})^2$$

Let us say

$$g(\theta) = \inf_{(\mu', \theta^{'}) \in C} \pi(\theta|\mu^{'}, y)$$
$$\Rightarrow \inf_{(\mu', \theta^{'}) \in C} IG\left(\frac{m-1}{2}, \frac{s^2}{2} + \frac{m}{2}(\mu^{'} - \bar{y})^2; \theta\right)$$

Here $IG(a, b, x)$ denotes density of Inverse gamma with parameter (a, b) and $x > 0$

The density of the inverse gamma given here can be written as

$$f(\theta) = k\theta^{-(\frac{m-1}{2}+1)} exp\left(\frac{1}{\theta}\left(-\frac{s^2}{2} - \frac{m}{2}(\mu^{'} - \bar{y})^2\right)\right)$$

Differentiating with Respect to $V = (\mu^{'}, \theta) = v$

$$\frac{df}{dv} = \frac{\left(\frac{m}{2} - \frac{1}{2}\right) m \left(\frac{mv}{2} + \frac{s^2}{2}\right)^{\frac{m}{2} - \frac{3}{2}} \mathrm{e}^{\frac{-\frac{mv}{2} - \frac{s^2}{2}}{\theta}}}{2\theta^{\frac{m+1}{2}} \Gamma(\frac{m-1}{2})} - \frac{m\theta^{-\frac{m+1}{2} - 1} \left(\frac{mv}{2} + \frac{s^2}{2}\right)^{\frac{m}{2} - \frac{1}{2}} \mathrm{e}^{\frac{-\frac{mv}{2} - \frac{s^2}{2}}{\theta}}}{2\Gamma(\frac{m-1}{2})}$$

Equating it to zero

$$\frac{\left(\frac{m}{2}-\frac{1}{2}\right)m\left(\frac{mv}{2}+\frac{s^2}{2}\right)^{\frac{m}{2}-\frac{3}{2}}\mathrm{e}^{\frac{-\frac{mv}{2}-\frac{s^2}{2}}{\theta}}}{2\theta^{\frac{m+1}{2}}\Gamma(\frac{m-1}{2})}=\frac{m\theta^{-\frac{m+1}{2}-1}\left(\frac{mv}{2}+\frac{s^2}{2}\right)^{\frac{m}{2}-\frac{1}{2}}\mathrm{e}^{\frac{-\frac{mv}{2}-\frac{s^2}{2}}{\theta}}}{2\Gamma(\frac{m-1}{2})}$$

$$\Rightarrow\frac{\left(\frac{m}{2}-\frac{1}{2}\right)m\left(\frac{mv}{2}+\frac{s^2}{2}\right)^{-1}\mathrm{e}^{\frac{-\frac{mv}{2}-\frac{s^2}{2}}{\theta}}}{\theta^{\frac{m+1}{2}}}=m\theta^{-\frac{m+1}{2}-1}\mathrm{e}^{\frac{-\frac{mv}{2}-\frac{s^2}{2}}{\theta}}$$

$$\Rightarrow\frac{\left(\frac{m}{2}-\frac{1}{2}\right)\left(\frac{mv}{2}+\frac{s^2}{2}\right)^{-1}}{\theta^{\frac{m+1}{2}}}=\theta^{-\frac{m+1}{2}-1}$$

$$\Rightarrow\left(\frac{m}{2}-\frac{1}{2}\right)\left(\frac{mv}{2}+\frac{s^2}{2}\right)^{-1}=\theta^{-\frac{m+1}{2}-1}\theta^{\frac{m+1}{2}}$$

$$\Rightarrow(m-1)\left(mv+s^2\right)^{-1}=\theta^{-1}$$

$$\Rightarrow\left(mv+s^2\right)^{-1}=\theta^{-1}(m-1)^{-1}$$

$$\Rightarrow\left(mv+s^2\right)=\theta\,(m-1)$$

$$\Rightarrow v=\theta-\frac{1}{m}(1+\frac{s^2}{m})$$

Let us say

$$v^*=\theta-\frac{1}{m}(1+\frac{s^2}{m})$$

Then we can say that

$$\frac{df(v)}{dv}|_{v<v^*}>0\;and\;\frac{df(v)}{dv}|_{v>v^*}<0$$

Hence the global minimal point in C is achieved by $f(v)$ when either $v=0$ or $v=d$ that is

$$\inf_{(\mu',\theta')}f(\theta)=min\left(IG\left(\frac{m-1}{2},\frac{s^2}{2};\theta\right),IG\left(\frac{m-1}{2},\frac{s^2}{2}+\frac{m}{2}d;\theta\right)\right)$$

Now we need to find the value of θ such that

$$IG\left(\frac{m-1}{2},\frac{s^2}{2};\theta\right)\leq IG\left(\frac{m-1}{2},\frac{s^2}{2}+\frac{m}{2}d;\theta\right)$$

$$\iff\frac{\frac{s^2}{2}^{\frac{m-1}{2}}}{\Gamma(\frac{m-1}{2})}\theta^{-(\frac{m-1}{2}+1)}exp\left(\frac{1}{\theta}\left(-\frac{s^2}{2}\right)\right)\leq\frac{(\frac{s^2}{2}+\frac{m}{2}d)^{\frac{m-1}{2}}}{\Gamma(\frac{m-1}{2})}\theta^{-(\frac{m-1}{2}+1)}exp\left(\frac{1}{\theta}\left(-\frac{s^2}{2}-\frac{m}{2}d\right)\right)$$

$$\iff\frac{s^2}{2}^{\frac{m-1}{2}}exp\left(\frac{1}{\theta}\left(-\frac{s^2}{2}\right)\right)\leq(\frac{s^2}{2}+\frac{m}{2}d)^{\frac{m-1}{2}}exp\left(\frac{1}{\theta}\left(-\frac{s^2}{2}-\frac{m}{2}d\right)\right)$$

$$\iff\frac{m-1}{2}log(\frac{s^2}{2})+\left(\frac{1}{\theta}\left(-\frac{s^2}{2}\right)\right)\leq\frac{m-1}{2}log(\frac{s^2}{2}+\frac{m}{2}d)+\left(\frac{1}{\theta}\left(-\frac{s^2}{2}-\frac{m}{2}d\right)\right)$$

$$\iff\left(\frac{1}{\theta}\left(-\frac{s^2}{2}\right)\right)-\left(\frac{1}{\theta}\left(-\frac{s^2}{2}-\frac{m}{2}d\right)\right)\leq\frac{m-1}{2}log(\frac{s^2}{2}+\frac{m}{2}d)-\frac{m-1}{2}log(\frac{s^2}{2})$$

$$\iff\left(\left(-\frac{s^2}{2\theta}\right)\right)+\left(\left(\frac{s^2}{2\theta}+\frac{m}{2\theta}d\right)\right)\leq\frac{m-1}{2}log(\frac{s^2}{2}+\frac{m}{2}d)-\frac{m-1}{2}log(\frac{s^2}{2})$$

$$\iff\frac{m}{2\theta}d\leq\frac{m-1}{2}log\left(1+\frac{md}{s^2}\right)$$

$$\iff\frac{md}{\theta}\leq(m-1)log\left(1+\frac{md}{s^2}\right)$$

$$\iff\theta\geq md\left[(m-1)log\left(1+\frac{md}{s^2}\right)\right]^{-1}$$

We define

$$\theta^*=md\left[(m-1)log\left(1+\frac{md}{s^2}\right)\right]^{-1}$$

so we can conclude

$$\inf_{(\mu',\theta')\in C} \pi(\theta|\mu^{'}, y) = \begin{cases} IG(\frac{m-1}{2}, \frac{s^2}{2} + \frac{md}{2}; \theta) \;\; if\, \theta < \theta^* \\ IG(\frac{m-1}{2}, \frac{s^2}{2}; \theta) \;\; if\, \theta \geq \theta^* \end{cases}$$