Generate Random numbers from the following densities

1.$f(x) = 2.5x\sqrt{x} \qquad 0 < x < 1$

First of all we will generaate 10 numbers from uniform distribution , we can use R for it using `runif()`. Now our generated numbers are

0.9943658, 0.2223798, 0.6928483, 0.7921491, 0.8973869, 0.4796866, 0.1116638, 0.759194, 0.6090344, 0.8692812

Now we will use **Inverse Transformation Method** For the random number generation

First of all we will calculate CDF of the density

$$\begin{aligned} F(x) &= \int_0^x f(x)dx \\ &= \int_0^x 2.5x\sqrt{x}dx \\ &= 2.5\int_0^x x^{\frac{3}{2}}dx \\ &= 2.5\left[\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right]_0^x \\ F(x) &= x^{\frac{5}{2}} \end{aligned}$$

Now,

$$\begin{aligned} u &= F(x) \\ \Rightarrow x &= F^{-1}(u) \\ \Rightarrow x &= u^{\frac{2}{5}} \end{aligned}$$

Now to get x we will put our random generated numbers from uniform distribution in the above equation

For u = 0.9943658, $x = (0.9943658)^{\frac{2}{5}}$, $x = 0.9977425$
For u = 0.2223798, $x = (0.2223798)^{\frac{2}{5}}$, $x = 0.5480726$
For u = 0.6928483, $x = (0.6928483)^{\frac{2}{5}}$, $x = 0.8634859$
For u = 0.7921491, $x = (0.7921491)^{\frac{2}{5}}$, $x = 0.9110092$
For u = 0.8973869, $x = (0.8973869)^{\frac{2}{5}}$, $x = 0.9576171$
For u = 0.4796866, $x = (0.4796866)^{\frac{2}{5}}$, $x = 0.7453891$
For u = 0.1116638, $x = (0.1116638)^{\frac{2}{5}}$, $x = 0.4160687$
For u = 0.759194, $x = (0.759194)^{\frac{2}{5}}$, $x = 0.8956557$
For u = 0.6090344, $x = (0.6090344)^{\frac{2}{5}}$, $x = 0.820081$
For u = 0.8692812, $x = (0.8692812)^{\frac{2}{5}}$, $x = 0.9455056$

2. $f(x) = \frac{x^2+x}{2} \qquad 0 \le x \le 1$

$$\begin{aligned} F(x) &= \frac{1}{2}x^2 + \frac{1}{2}x \\ F(x) &= p_1F_1(x) + p_2F_2(x) \\ Taking\ \ p_1 &= p_2 = \frac{1}{2} \\ Set\ \ F_1(x) &= u\ and\ F_2(x) = u \\ x &= u^{\frac{1}{2}}\ and\ x = u \end{aligned}$$

Now Generate two random numbers u_1 and u_2 from $U(0,1)$

Now if $u_1 \leq \frac{1}{2}$ set $x = \sqrt{u_2}$
otherwise set $x = u_2$

3. Beta(2,4)

We will use **Acceptance/Rejection** Method. The density function of Beta is given by

$$f(x;a,b) = \frac{x^{a-1}(1-x)^{b-1}}{\beta(a,b)}$$

Now by taking a = 2 and b = 4 we get

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \Rightarrow \beta(a,b) = \frac{\Gamma(2)\Gamma(4)}{\Gamma(2+6)} = \frac{1}{20}$$

So the density will be given by

$$f(x;2,4) = 20x(1-x)^3$$

Now let g(x) = 1 ∀ x ∈ (0,1)

$$c = \sup_x \left\{ \frac{f(x)}{g(x)} \right\} \Rightarrow \left\{ \frac{f(x)}{g(x)} \right\}' = 0$$

That gives us

$$\{20x(1-x)^3\}' = 0$$
$$(1-x)^3 + 3x(1-x)^2(-1) = 0$$
$$x = \frac{1}{4} = 0.25$$

So

$$c = \frac{f(0.25)}{g(0.25)} = 20 \cdot 0.25 \cdot (1-0.25)^3 = 2.109375$$

Now generate two random numbers u_1 and u_2 from U(0,1) , now if $u_2 \leq \frac{f(u_1)}{c \cdot g(u_1)}$ then accept u_1 as a random number generated from $\beta(2,4)$, otherwise repeat again

4. $Gamma(\frac{5}{2},1)$

We will use **Acceptance/Rejection** Method.

The density function of gamma distribution is given by

$$f(x) = \frac{e^{-x}x^{\frac{5}{2}-1}}{\Gamma\left(\frac{5}{2}\right)}$$

Where $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$

So

$$f(x) = \frac{4}{3\sqrt{\pi}} e^{-x} x^{\frac{3}{2}}$$

Let us take $g(x) \sim exp(\lambda = 2/5)$ then density of g(x) is given by

$$g(x) = \frac{2}{5}e^{-\frac{2}{5}x}$$

Then

$$c = \sup_x \left\{ \frac{f(x)}{g(x)} \right\} \Rightarrow \left\{ \frac{f(x)}{g(x)} \right\}' = 0$$

So,

$$\left\{ \frac{\frac{4}{3\sqrt{\pi}}e^{-x}x^{\frac{3}{2}}}{\frac{2}{5}e^{-\frac{2}{5}x}} \right\}' = 0$$

$$\left\{ e^{-\frac{3}{5}x}x^{\frac{3}{2}} \right\}' = 0$$

$$\left\{ e^{-\frac{3}{5}x}\frac{-5}{3}x^{\frac{3}{2}} + \frac{3}{2}x^{\frac{1}{2}}e^{-\frac{3}{5}x} \right\} = 0$$

$$\frac{3}{2}x^{\frac{1}{2}} = \frac{5}{3}x^{\frac{3}{2}}$$

$$x = \frac{2/3}{5/3} = \frac{2}{5}$$

Now

$$c = \frac{f(\frac{5}{2})}{g(\frac{5}{2})} = \frac{\frac{4}{3\sqrt{\pi}}e^{-\frac{5}{2}}\frac{5}{2}^{\frac{3}{2}}}{\frac{2}{5}e^{-\frac{2}{5}\cdot\frac{5}{2}}} = 1.659136$$

Now generate two random numbers u_1 from $exp\left(\frac{2}{5}\right)$ and u_2 from U(0,1) , now if $u_2 \leq \frac{f(u_1)}{c \cdot g(u_1)}$ then accept u_1 as a random number generated from $\beta(2,4)$, otherwise repeat again

5. $Normal(\mu, \sigma^2)$

We will solve this using **Box Muller Method**

In this method we will generate two random variables with $N(0,1)$ and then us transformation to generate random variable from $N(\mu, \sigma^2)$

Following are the steps

1. Generate two random variables u_1 and u_2 from Standard Uniform Distribution
2. Then set

$$X_1 = \sqrt{-2log(U_1)}cos(2\pi U_2)$$

$$X_2 = \sqrt{-2log(U_1)}sin(2\pi U_2)$$

3. Now X_1 and X_2 are independent standard normal variates , now we can transform using following equation

$$Y_1 = X_1\sigma + \mu$$

$$Y_2 = X_2\sigma + \mu$$

Now Y_1 and Y_2 are distributed

$$Y_1, Y_2 \sim N(\mu, \sigma^2)$$